ЕДНА ЗАДАЧА НА ГУРСА ЗА УРАВНЕНИЕТО "СИНУС-ГОРДОН"

Дафинка Цокова Ангелова[1], Любомир Петров Георгиев[2], Васил Георгиев Ангелов[3]

Висше строително училище "Л. Каравелов" – София,
Минно-геоложки университет "Св. Ив. Рилски" – София

A GOURSAT PROBLEM FOR SINE-GORDON EQUATION

Dafinka Tzokova Angelova, Ljubomir Petrov Georgiev, Vasil Georgiev Angelov

"L. Karavelov" Civil Engineering Higher School – Sofia,
University of Mining and Geology "St. I. Rilski" – Sofia

Abstract: *Here we present a different approach for solving sine-Gordon equation. We obtain a unique continuous solution for the Goursat problem with continuous differentiable initial and boundary functions as a fixed point of a suitably chosen operator. This solution can be achieved by successive approximations.*

Key words: *Sine-Gordon equation, Goursat problem, Fixed point method.*

1. Introduction

The one dimensional sine-Gordon equation, which has the general form

$$u_{tt} - u_{xx} + \sin u = 0,\ L_1 < x < L_2,\ t > t_0, \tag{1}$$

with $u=u(x,t)$ a sufficiently differentiable function, has been investigated theoretically by mathematicians in the 19th century like Enneper, Eisenhart, Darboux, Bianchi etc (see for example [14]). The term "sine-Gordon equation" is presumably a kind of joke, obviously originating in the name of the "Klein-Gordon equation" in relativistic field theories. The name was suggested by M. Kruskal, in analogy with the linear Klein-Gordon equation (where u appears in place of sin u). The equation grew greatly in importance when it was realized that it led to solutions ("kink" and "antikink") with the collisional properties of solitons ([9], [11, [26], [30], [32] etc). The sine-Gordon equation, too, has a wide range of applications in physics, not only in relativistic field theories but in solid-state physics, dislocation theory of crystals, propagation in ferromagnetic materials of waves carrying rotations of the direction of magnetization, laser pulses in two state media, nonlinear optics, including the propagation of fluxions in Josephson junctions (a junction between two superconductors), the motion of rigid pendula attached to a stretched wire etc (see [4] - [6], [12], [17], [19], [29]).

The Bäcklund transforms of surfaces [3] can be viewed as a transformation, introduced by Bäcklund, for finding of the exact solutions of the sine-Gordon equation. The initial value problem for (1) is solved by the inverse-scattering method (see [1]). The periodic problem is

[1] доц. д-р Дафинка Цокова Ангелова, катедра „Математика и физика", ВСУ „Л. Каравелов", 1373 София, ул. Суходолска 175, email: angelova@vsu.bg
[2] гл. ас. Любомир Петров Георгиев, катедра „Математика", МГУ „Св. Ив. Рилски", 1700 София, Студентски град, кв. Дървеница, e-mail: lubo_62@abv.bg
[3] проф. дтн Васил Георгиев Ангелов, катедра „Математика", МГУ „Св. Ив. Рилски", 1700 София, Студентски град, кв. Дървеница, e-mail: angelov@mgu.bg

studied by means of an algebraic-geometric method (similar to the case of the Korteweg–de Vries equation) in [16]. In particular, there the authors obtain explicit expressions for the finite-gap solutions of the equation in terms of θ-functions on the corresponding Abelian varieties. Almost-periodic solutions of (1) are studied by Kozel and Kotlyarov [22]. Exact (solitory and traveling) solutions have been obtained by Yang [31], and for its modification – by Kobayashi and Izutsu [21], Saermark [28], Leibrandt [23] and etc. Equation (1) has been also analyzed numerically by means of finite-difference (FDM), finite elements, pseudospectral, Adomian decomposition method etc. As far as FDM can be found, among other publications, in Guo, Pascual, Rodriguez and Vazkuez [18], Ramos [27], Bratsos [7], [8].

The main purpose of the present note is to obtain an approximation of the exact solution. It is obtained in a few steps and so close to the exact solution as $\lambda > 0$ is sufficiently great.

The existence of unique global solution of two initial-boundary value problems for the equation (1) (as $L_1 = 0, L_2 = \infty,\ t \in (0,T)$), with sufficiently smooth initial data $u(x,0),\, u_t(x,0)$ and boundary conditions $u(0,t) = const$, or $u_x(0,t) + a\cos\frac{u(0,t)}{2} + b\sin\frac{u(0,t)}{2} = 0, a, b \in \Re$, respectively, is proved by Fokas [15].

Positive solutions of the elliptic sine-Gordon equation

$$-\Delta u = \lambda \sin u,\ \lambda = const \ge 0$$

on bounded domain with homogenous Dirichlet boundary conditions, which models the steady state of the Josephson π-junction in superconductity, is studied by Ding, Chen and Li [13].

The so-called double sine-Gordon equation is given by

$$u_{xt} \pm \left(\sin u + \eta \sin\frac{u}{2} \right) = 0 \tag{2}$$

(Calogero and Degasperis [11]). Exact, multiple soliton solutions of (2) are obtained in [10].

The equation (2) can be transformed in the form (see [19])

$$\Lambda_{xy} = \sin \Lambda . \tag{3}$$

The equivalence of the methods for solving both the Goursat problem and the Cauchy problem for the sine-Gordon equation (3) is discussed by Kaup and Newell [20]. Sufficient conditions on the initial data for which each problem may be solved by the inverse scattering transform are given.

The sine-Gordon equation (3) in light cone coordinates is solved by Leon [24] when Dirichlet conditions on the L-shape boundaries of the quarter plane (x,t) are prescribed in a class of functions that vanish at infinity at initial time. The method is based on the inverse spectral transform (IST) (see also [32]) for the Schrödinger spectral problem on the semi-line $x>0$ solved as a Hilbert boundary value problem.

Pelloni [25] use the Fokas transorm method too to analyze the boundary value problems of the sine-Gordon equation on a finite interval and she has given a representaion of the solution of such problems for the equation $q_{xt} + \sin q = 0,\ (x,t) \in (0,L) \times (0,T)$, with an arbitrary $L > 0$. In particular, she solved a boundary value problem with one initial and one constant boundary condition ($q(0,t) = \gamma = const, t > 0$).

2. Main result

Here we consider the Goursat problem for the sine-Gordon equation

$$\begin{cases} \Lambda_{xy} = \sin \Lambda,\ (x,y) \in (0,\infty) \times (0,\infty) \\ \Lambda(x,\,0) = f(x),\ x \in [0,\infty) \\ \Lambda(0,y) = g(y),\ y \in [0,\infty) \end{cases}, \tag{4}$$

where $\Lambda = \Lambda(x, y)$ is the unknown continuous function and $\Lambda_{xy} = \dfrac{\partial^2 \Lambda}{\partial x \partial y}$ is its second order mixed partial derivative.

Here we suppose, the "initial" function $g(y)$ and the "boundary" function $f(x)$ belong to the set $C([0,\infty)) \cap C^1((0,\infty))$, consisting of all continuous functions on $[0,\infty)$, which are continuously differentiable on $(0,\infty)$. We suppose, for consistency, $f(0) = g(0)$.

Our main purpose is to obtain not only the existence and uniqueness of a solution of the equation (3) but to present an approximately solution, obtained easily step by step.

Integrating (3) on the first argument from 0 to x and on the second argument from 0 to y (without meaning the order of integration), (4) transforms to the initial-boundary value problem for existing of continuous solution of the integral equation

$$\Lambda(x, y) = f(x) + g(y) - a + \int_0^x \int_0^y \sin \Lambda(\xi, \eta) d\xi d\eta, \ (x, y) \in (0,\infty) \times (0,\infty) , \tag{5}$$

(with the constant $a = f(0) = g(0)$), satisfying the initial-boundary conditions

$$\begin{cases} \Lambda(x, 0) = f(x), \ x \in [0,\infty) \\ \Lambda(0, y) = g(y), \ y \in [0,\infty) \end{cases} . \tag{6}$$

(Certainly, if the function Λ is a continuous solution of (5), (6), it has a continuous second order mixed partial derivative and that function Λ is a solution of (4).)

Denote by X the set of all continuous on $[0,\infty) \times [0,\infty)$ functions, which are equal to $f(x)$ on $[0,\infty) \times \{0\}$ and equal to $g(y)$ on $\{0\} \times [0,\infty)$. The set X is not empty – the function

$$\Lambda_0(x, y) = \begin{cases} f(x) + g(y) - a, & (x, y) \in (0,\infty) \times (0,\infty) \\ f(x), & y = 0, x \in [0,\infty) \\ g(y), & x = 0, y \in [0,\infty) \end{cases}$$

belongs to X.

Let the constant λ from the interval $(1,\infty)$ be arbitrary and fixed.

Denote by Ψ the set of all compacts K: $K \subset [0,\infty) \times [0,\infty)$.

Introduce the family of the pseudometrics $\rho_K : X \times X \to [0,\infty)$,

$$\rho_K(\varphi, \psi) = \max\left\{ e^{-\lambda(x+y)} |\varphi(x, y) - \psi(x, y)| : (x, y) \in K \right\}.$$

This family defines a uniformity on X, indexed by the elements of the set $\Im = \{\rho_K(\varphi, \psi) : K \in \Psi\}$. So we obtain the uniform space $(X, \Im)$.

Define the mapping $j : \Psi \to \Psi$, $j(K) = K_x \times K_y, \ \forall K \in \Psi$, where

$$K_x = [0, \max\{x : (x, y) \in K\}], \ K_y = [0, \max\{y : (x, y) \in K\}],$$

and the operator T: $\Lambda \in X \to T(\Lambda)$, acting by the formula

$$T(\Lambda)(x, y) = \begin{cases} f(x) + g(y) - a + \int_0^x \int_0^y \sin \Lambda(\xi, \eta) d\xi d\eta, & (x, y) \in (0,\infty) \times (0,\infty) \\ f(x), & y = 0, x \in [0,\infty) \\ g(y), & x = 0, y \in [0,\infty) \end{cases} . \tag{7}$$

We prove, the operator $T : X \to X$ has a unique fixed point (thus, the problem (5), (6) has a unique continuous solution, which, in particular, is a unique continuous solution for the Goursat problem (4)), using the fixed point techniques for Φ-contractive mappings in uniform spaces (see [2]).

It is easy to check

$$j^2(K) = j\big(j(K)\big) = j(K) \Rightarrow j^n(K) = j(K),\ K \in \Psi,\ n = 2,3,..., \tag{8}$$

and $T(\Lambda)(x,y) - T(\Lambda)(x_0,y_0) =$

$$= f(x) - f(x_0) + g(y) - g(y_0) + \int_{x_0}^{x}\int_{0}^{y_0} \sin\Lambda(\xi,\eta)d\xi d\eta + \int_{x_0}^{x}\int_{y_0}^{y} \sin\Lambda(\xi,\eta)d\xi d\eta + \int_{0}^{x_0}\int_{y_0}^{y} \sin\Lambda(\xi,\eta)d\xi d\eta .$$

Hence $T(\Lambda)(x,y)$ is continuous function of (x, y). Having in mind the definition (7) of the operator T we conclude $T(\Lambda) \in X$ for any $\Lambda \in X$.

By the definition of the mapping $j:\Psi \to \Psi$ and in view of (8), we have

$$\rho_K(\varphi,\psi) \le \rho_{j(K)}(\varphi,\psi) = \rho_{j^n(K)}(\varphi,\psi),\ n = 2,3,...,$$

and then we obtain

$$\rho_{j^n(K)}(\varphi,\psi) \le q(K,\varphi,\psi) = \rho_{j(K)}(\varphi,\psi) < \infty \text{ for } (\varphi,\psi) \in X \times X,\ K \in \Psi,\ n = 0,1,2,...$$

In other words, the set X is j-bounded, and hence, if the operator $T: X \to X$ has at least one fixed point, it will be the unique one (see [2]).

For the function Λ_0, in particular, we have:

$$\rho_{j^n(K)}(\Lambda_0, T(\Lambda_0)) \le \rho_{j(K)}(\Lambda_0, T(\Lambda_0)) = q(K,\Lambda_0),\ n = 0,1,2,...$$

Moreover, for $(x,y) \in K, x > 0, y > 0$

$$\left|\Lambda_0(x,y) - T(\Lambda_0)(x,y)\right| = \left|\int_0^x\int_0^y \sin\Lambda_0(\xi,\eta)d\xi d\eta\right| \le xy$$

, and then

$$e^{-\lambda(x+y)}\left|\Lambda_0(x,y) - T(\Lambda_0)(x,y)\right| \le \left[\max\left\{ze^{-\lambda z} : z \in [0,\infty)\right\}\right]^2 = (e\lambda)^{-2},\ \text{i. e. } q(K,\Lambda_0) \le (e\lambda)^{-2} < \infty .$$

Let $\Lambda, \overline{\Lambda}$ be two arbitrary and fixed functions, belonging to X, K – a fixed compact from Ψ and $(x,y) \in K$. If at least one of the coordinates (x,y) is 0, the difference $T(\Lambda) - T(\overline{\Lambda})$ is equal to 0 in this point, in view of the definition of T. For $x > 0, y > 0$ we get:

$$\left|T(\Lambda)(x,y) - T(\overline{\Lambda})(x,y)\right| = \left|\int_0^x\int_0^y \left(\sin\Lambda(\xi,\eta) - \sin\overline{\Lambda}(\xi,\eta)\right)d\xi d\eta\right| =$$

$$= \left|\int_0^x\int_0^y 2\sin\frac{\Lambda(\xi,\eta) - \overline{\Lambda}(\xi,\eta)}{2}\cos\frac{\Lambda(\xi,\eta) + \overline{\Lambda}(\xi,\eta)}{2}d\xi d\eta\right| \le$$

$$\le \int_0^x\int_0^y 2\left|\sin\frac{\Lambda(\xi,\eta) - \overline{\Lambda}(\xi,\eta)}{2}\right|d\xi d\eta \le \int_0^x\int_0^y \left|\Lambda(\xi,\eta) - \overline{\Lambda}(\xi,\eta)\right|d\xi d\eta \le$$

$$\le \rho_{j(K)}\left(\Lambda,\overline{\Lambda}\right)\int_0^x e^{\lambda x}d\xi\int_0^y e^{\lambda y}d\eta = \rho_{j(K)}\left(\Lambda,\overline{\Lambda}\right)\frac{e^{\lambda x}-1}{\lambda}\cdot\frac{e^{\lambda y}-1}{\lambda} \le \frac{\rho_{j(K)}\left(\Lambda,\overline{\Lambda}\right)}{\lambda^2}e^{\lambda(x+y)},$$

which follows

$$\rho_K\left(T(\Lambda), T(\overline{\Lambda})\right) \le \frac{\rho_{j(K)}\left(\Lambda,\overline{\Lambda}\right)}{\lambda^2}. \tag{9}$$

Now introduce the linear function $\Phi:[0,\infty) \to [0,\infty)$: $\Phi(t) = t\lambda^{-2}, \lambda = const > 1.$

It is easy to see $\Phi(t) < t$ for any $t > 0$ and the function $\dfrac{\Phi(t)}{t} = \lambda^{-2} = const$ is non-decreasing.

Via (9) it follows T is Φ-contractive operator in the uniform space $(X, \Im)$.

Since all conditions of the theorems in [2] (with the special family of functions $\{\Phi_K : K \in \Psi\}$ _ $\Phi_K \equiv \Phi, \forall K \in \Psi$!) – for existence and uniqueness of a fixed point, hold good, we conclude the problem (5), (6) (and consequently (4)) has a unique continuous solution.

In particular, the fixed point of T can be obtained as the limit in the uniform space $(X, \Im)$ of the following functional sequence $\{\Lambda_n\}_{n=0}^{\infty}$: $\Lambda_n = T(\Lambda_{n-1}),\ n = 1,2,...$,

$$\Lambda_0(x,y) = \begin{cases} f(x) + g(y) - a, & (x,y) \in (0,\infty)\times(0,\infty) \\ f(x), & y = 0, x \in [0,\infty) \\ g(y), & x = 0, y \in [0,\infty) \end{cases}$$

i.e.
$$\Lambda_1(x,y) = \Lambda_0(x,y) + \int_0^x\int_0^y \sin\Lambda_0(\xi,\eta)d\xi d\eta = f(x) + g(y) - a + \int_0^x\int_0^y \sin\big(f(\xi) + g(\eta) - a\big)d\xi d\eta,$$

$$\Lambda_2(x,y) = \Lambda_0(x,y) + \int_0^x\int_0^y \sin\Lambda_1(\xi,\eta)d\xi d\eta, ...,\ \Lambda_n(x,y) = \Lambda_0(x,y) + \int_0^x\int_0^y \sin\Lambda_{n-1}(\xi,\eta)d\xi d\eta, ...$$

For any fixed compact $K \in \Psi$ the functions from the above sequence belong to the set $B_\delta(\Lambda_0) = \{\Lambda \in X : \rho_K(\Lambda, \Lambda_0) < \delta\}$ with $\delta = e^{-2}\left(\lambda^2 - 1\right)^{-1}$.

Indeed, via (9) we obtain

$$\rho_K\left(\Lambda_{m+1}, \Lambda_m\right) \le \alpha\rho_{j(K)}\left(\Lambda_m, \Lambda_{m-1}\right) \le ... \le \alpha^m \rho_{j(K)}\left(\Lambda_1, \Lambda_0\right),\ m = 1,2,...,$$

where $\alpha = \lambda^{-2}$. Then:

$$\rho_K\left(\Lambda_n, \Lambda_0\right) \le \sum_{m=0}^{n-1}\rho_K\left(\Lambda_{m+1}, \Lambda_m\right) \le \sum_{m=0}^{n-1}\alpha^m \rho_{j(K)}\left(\Lambda_1, \Lambda_0\right) = \rho_{j(K)}\left(\Lambda_1, \Lambda_0\right)\sum_{m=0}^{n-1}\alpha^m,\ n = 1,2,...$$

Since $\rho_{j^s(K)}(\Lambda_1, \Lambda_0) = \rho_{j^s(K)}(\Lambda_0, T(\Lambda_0)) = q(K, \Lambda_0) \le \alpha e^{-2}\ (s = 0,1)$, the following estimate is hold good for $n \ge 1$: $\rho_K\left(\Lambda_n, \Lambda_0\right) \le e^{-2}\alpha\dfrac{1-\alpha^n}{1-\alpha} < e^{-2}\dfrac{\alpha}{1-\alpha} = e^{-2}\dfrac{\lambda^{-2}}{1-\lambda^{-2}} = \delta$.

3. Conclusions

In the present paper a new approach for investigation of the sine-Gordon equation is used – a fixed point method in uniform space. This approach can be used successfully to establish sufficient conditions for the existence of positive and bounded solutions, also and conditions for oscillation of solutions of (3).

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